22 The Existence of Parallel Lines

The concept of parallel lines has led to both the most fruitful and the most frustrating developments in plane geometry. Euclid (c. 330-275 B.C.E.) defined two segments to be parallel if no matter how far they are extended in both directions, they never meet.

The history of the parallel postulate is fascinating. In fact, many mathematicians attempted to prove the Fifth Postulate, and some thought they had succeeded.

We shall see in this chapter that the axioms which we have adopted so far (and which are a refinement of those of Euclid) are sufficient only for proving the existence of parallel lines, but not the uniqueness.

Definition. **(transversal).** Given three distinct lines ℓ, ℓ_1, and ℓ_2, we say that ℓ is a transversal of ℓ_1 and ℓ_2 if ℓ intersects both ℓ_1 and ℓ_2, but in different points.

Definition. **(alternate interior angles, corresponding angles).** Assume that the line $\overleftrightarrow{GH}$ is transversal to $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$ in a metric geometry and that $\overleftrightarrow{AC} \cap \overleftrightarrow{GH} = \{B\}$ and $\overleftrightarrow{DF} \cap \overleftrightarrow{GH} = \{E\}$. If the points A, B, C, D, E, F, G and H are situated in such a way that

(i) $A-B-C$, $D-E-F$, and $G-B-E-H$, and

(ii) A and D are on the same side of $\overleftrightarrow{GH}$

then $\measuredangle ABE$ and $\measuredangle FEB$ are a pair of alternate interior angles and $\measuredangle ABG$ and $\measuredangle DEB$ are a pair of corresponding angles.

Theorem. Let ℓ_1 and ℓ_2 be two lines in a neutral geometry. If there is a transversal ℓ of ℓ_1 and ℓ_2 with a pair of alternate interior angles congruent then there is a line ℓ' which is perpendicular to both ℓ_1 and ℓ_2.

1. Prove the above Theorem. [Th 7.1.1, p171]

Theorem. In a neutral geometry, if ℓ_1 and ℓ_2 have a common perpendicular, then ℓ_1 is parallel to ℓ_2. In particular, if there is a transversal to ℓ_1 and ℓ_2 with alternate interior angles congruent, then $\ell_1 \| \ell_2$.

2. Prove the above Theorem. [Th 7.1.2, p172]

By above theorem, if ℓ_1 and ℓ_2 have a common perpendicular then $\ell_1 \| \ell_2$. Is the converse true: If $\ell_1 \| \ell_2$, do ℓ_1 and ℓ_2 have a common perpendicular?

3. In the Poincaré Plane let $\ell = {}_0L$ and $\ell' = {}_1L_1$. Show that $\ell \| \ell'$ but that there is no line perpendicular to both ℓ and ℓ'. [Ex 7.1.3, p173]

Theorem. In a neutral geometry, let ℓ be a line and $P \notin \ell$. Then there is a line ℓ' through P which is parallel to ℓ.

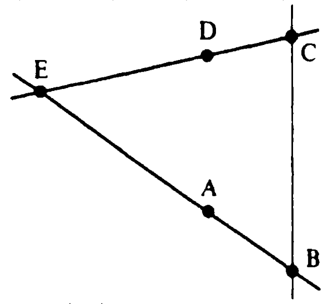
4. Prove the above Theorem. [Th 7.1.4, p173]

5. Show that in the Poincaré Plane there is more than one line through $P(3,4)$ which is parallel to ${}_{-5}L$. [Ex 7.1.5, p174]

Definition. **(Euclid's Fifth Postulate (EFP))** A protractor geometr satisfies Euclid's Fifth Postulate (EFP) if whenever $\overleftrightarrow{BC}$ is a transversal of $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$ with

(i) A and D on the same side of $\overleftrightarrow{BC}$

(ii) $m(\measuredangle ABC) + m(\measuredangle BCD) < 180$



then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point E on the same side of $\overleftrightarrow{BC}$ as A and D.

Theorem. If ℓ is a line and $P \notin \ell$ in a neutral geometry which satisfies EFP, then there exists a unique line ℓ' through P which is parallel to ℓ.

6. Prove the above Theorem. [Th 7.1.6, p175]

Definition. **(Euclidean Parallel Property (EPP))** An incidence geometry satisfies the Euclidean Parallel Property (EPP) if for every line ℓ and every point P, there is a unique line through P which is parallel to ℓ.

Note that EPP is a property of an incidence geometry so that the Taxicab Plane, Euclidean Plane, and $\mathbb{R}^2$ with the max distance all satisfy EPP because they all have the same underlying incidence geometry, and it satisfies EPP. Of course, only the second is a neutral geometry.

Theorem. If a neutral geometry satisfies EPP then it also satisfies EFP.

7. Prove the above Theorem. [Th 7.1.7, p176]

8. In a neutral geometry if $\measuredangle ABC$ is acute then the foot of the perpendicular from A to $\overleftrightarrow{BC}$ is an element of int($\overrightarrow{BC}$).

9. Given two lines and a transversal in a protractor geometry, prove that a pair of alternate interior angles are congruent if and only if a pair of corresponding angles are congruent.

10. In a neutral geometry, if ℓ is a transversal of ℓ_1 and ℓ_2 with a pair corresponding angles congruent, prove that $\ell_1 \| \ell_2$.

11. In a neutral geometry, if $\overleftrightarrow{BC}$ is a common perpendicular of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, prove that if ℓ is a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ that contains the midpoint of $\overline{BC}$ then a pair of alternate interior angles for ℓ are congruent.

23 Saccheri Quadrilaterals

In 1733 there appeared the book *Euclid Vindicated of All Flaw* by the Jesuit priest Gerolamo Saccheri. In it the author purported to prove Euclid's Fifth Postulate as a theorem. We now recognize basic flaws in his argument at certain crucial steps. However, the book was and is important in the development of the theory of parallels because it was the first to investigate the consequences of assuming the negation of Euclid's Fifth Postulate.

Despite his failure to actually prove Euclid's Postulate as a theorem, Saccheri did contribute a substantial body of correct results. Did he know about the flaws in his proof? Certainly the erroneous proofs were unlike any of the rest of his carefully reasoned development. It has been suggested that Saccheri knew what he did was fallacious and that the "proof" was included so that the Church would approve the publication of his work.

Definition. **(Saccheri quadrilateral, lower base, upper base, lower base angles, upper base angles)** A quadrilateral $\square ABCD$ in a protractor geometry is a Saccheri quadrilateral if $\measuredangle A$ and $\measuredangle D$ are right angles and $\overline{AB} \cong \overline{CD}$. In this case we write ⓢ$ABCD$. The lower base of ⓢ$ABCD$ is $\overline{AD}$, the upper base is $\overline{BC}$, the legs are $\overline{AB}$ and $\overline{CD}$, the lower base angles are $\measuredangle A$ and $\measuredangle D$, and the upper base angles are $\measuredangle B$ and $\measuredangle C$.

The basic approach of Saccheri (and those who followed him) was to try to prove something which turned out not to be true: that every Saccheri quadrilateral was actually a rectangle. If that were true it would not be hard to prove that EPP holds.

Theorem. In a neutral geometry a Saccheri quadrilateral ⓢ$ABCD$ is a convex quadrilateral.

Definition. **(congruent convex quadrilaterals)** Two convex quadrilaterals in a protractor geometry are congruent if the corresponding sides and angles are congruent. In this case we write ⓢ$ABCD \cong$ ⓢ$EFGH$.

Theorem. In a neutral geometry, if $\overline{AD} \cong \overline{PS}$ and $\overline{AB} = \overline{PQ}$, then ⓢ$ABCD \cong$ ⓢ$PQRS$.

Corollary In a neutral geometry if ⓢ$ABCD$ is a Saccheri quadrilateral then ⓢ$ABCD \cong$ ⓢ$DCBA$ and $\measuredangle B \cong \measuredangle C$.

Theorem. **(Polygon Inequality).** Suppose $n > 3$. If P_1, P_2, ..., P_n, are points in a neutral geometry then

$$d(P_1, P_n) \leq d(P_1, P_2) + d(P_2, P_3) + \ldots + d(P_{n-1}, P_n).$$

Theorem. In a neutral geometry, given ⓢ$ABCD$, then $\overline{BC} > \overline{AD}$.

Theorem. In a neutral geometry, given ⓢ$ABCD$, then $\measuredangle ABD \leq \measuredangle BDC$.

Theorem. In a neutral geometry the sum of the measures of the acute angles of a right triangle is less than or equal to 90.

Theorem. **(Sacchcri's Theorem).** In a neutral geometry, the sum of the measures of the angles of a triangle is less than or equal to 180.

It must be remembered that above theorem is the best possible result. We have already seen an example of a triangle in $\mathcal{H}$ in which the sum of the measures of the angles is actually strictly less than 180. In your high school geometry course you learned that the sum of the measures of the angles of a triangle was exactly 180. That result was correct because you were dealing exclusively with a geometry which satisfied EPP.

...for the rest of results and lots of interesting problems, see the lectures and the given book, pages 178-187...

(#) U neutralnoj geometriji ako je $\measuredangle ABC$ oštar ugao pokazati da je tada podnožje okomice iz tačke A na pravu $p(B,C) = \overleftrightarrow{BC}$ element $int(\overrightarrow{BC})$.

Rj.

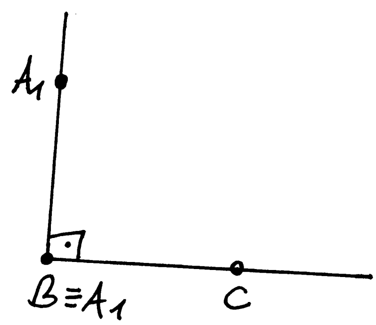
Označimo sa A_1 tačku na $\overleftrightarrow{BC}$ tako da je $AA_1 \perp \overleftrightarrow{BC}$

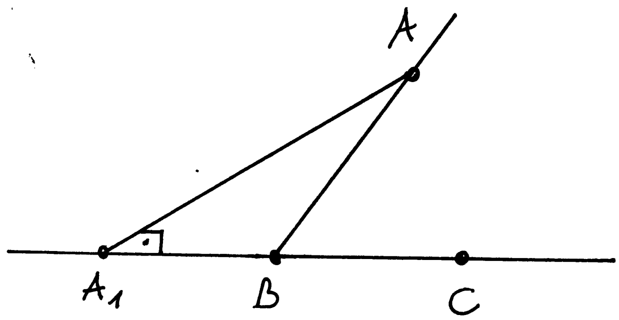
Za tačku A_1 je moguć jedan od sljedeća tri slučaja

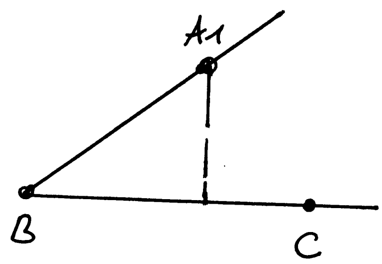
1° $A_1 - B - C$

2° $A_1 = B$

3° $A_1 \in int(\overrightarrow{BC})$







Pokažimo da slučajevi 1° i 2° nisu mogući.

Ako bi bio prvi slučaj, primjetimo da je tada ΔAA_1B pravougli trougao sa pravim uglom $\measuredangle AA_1B$. S druge strane $\measuredangle ABC$ je vanjski ugao ΔAA_1B i vrijedi: $\measuredangle ABC > \measuredangle AA_1B$

#kontradikcija
($\measuredangle ABC$ je oštar ugao)

Da nije moguć drugi slučaj ostavljamo ZA VJEŽBU

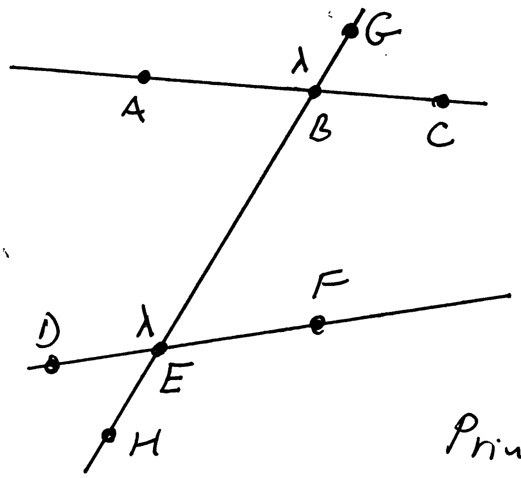
Prema tome mora vrijediti da je $A_1 \in int(\overrightarrow{BC})$

q.e.d.

(#) Date su dvije prave i transferzala u protractor geometriji. Pokazati da je par naizmjeničnih unutrašnjih uglova podudaran ako i samo ako je par saglasnih uglova podudaran.

Rj:

„⇐" Pretpostavimo da je par saglasnih uglova podudaran.



Uvedimo oznake kao na slici ($A-B-C$, $D-E-F$, $G-B-E-H$, A i D su sa iste strane prave $\overleftrightarrow{BE}$). Prema pretpostavci $\measuredangle ABG \cong \measuredangle DEB$. Označimo mjeru ovih uglova sa λ.

Primjetimo da uglovi $\measuredangle ABG$ i $\measuredangle ABE$ formiraju linearan par. Prema teoremu linearnog para imamo da su ova dva ugla suplementarna tj. $m(\measuredangle ABE) = 180 - \lambda$. ...(1)

Slično imamo za uglove $\measuredangle DEB$ i $\measuredangle BEF$. Oni su suplementarni pa je $m(\measuredangle BEF) = 180 - \lambda$ ...(2)

(1) i (2) $\Rightarrow$ $\measuredangle ABE \cong \measuredangle FEB$

$\Downarrow$

par naizmjeničnih unutrašnjih uglova je podudaran

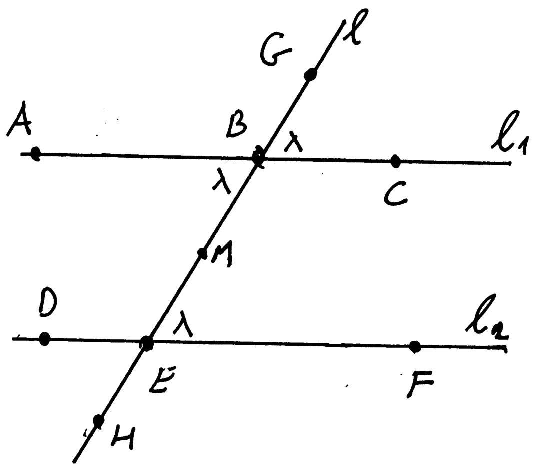
q.e.d.

„⇒" Pretpostavimo da je par naizmjeničnih unutrašnjih uglova podudaran i pokažimo da je tada par saglasnih uglova podudaran.

ZAVRŠITI ZA VJEŽBU

(#) U neutralnoj geometriji, ako je l transferzala pravih l_1 i l_2 sa podudarnim parom saglasnih uglova, dokazati da je tada $l_1 \parallel l_2$.

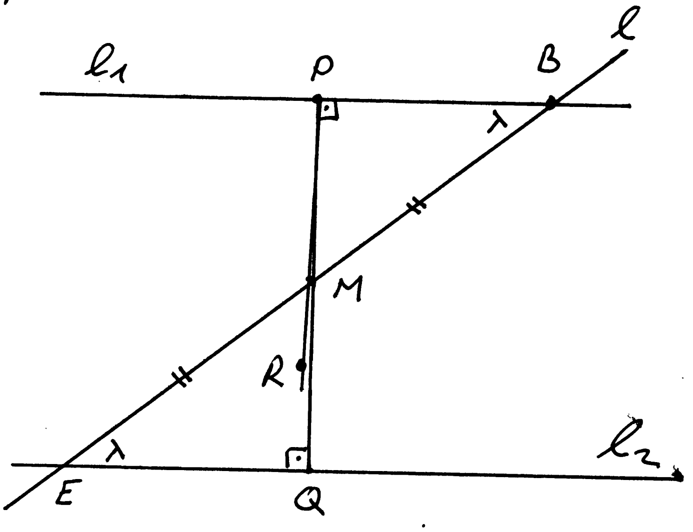
Rj.



Uredimo oznake kao na slici ($A-B-C$, $D-E-F$, $G-B-E-H$ i A i D su sa iste strane prave l)

Prema pretpostavci $\measuredangle GBC \cong \measuredangle BEF$ (mjere ovih uglova označimo sa λ)

Pretpostavimo da su $\measuredangle GBC$ i $\measuredangle BEF$ oštri uglovi i neka je M središte duži $\overline{BE}$. Kako su uglovi $\measuredangle ABE$ i $\measuredangle GBC$ unakrsni to je $m(\measuredangle ABM) = \lambda$. Označimo sa P i Q redom okomice iz tačke M na prave l_1 i l_2. S obzirom da su $\measuredangle ABM$ i $\measuredangle FEM$ oštri to su P i A sa iste strane prave l, a F i Q su iste strane prave l. Time su tačke P i Q sa različitih strana prave l (ZAŠTO?)



Želimo pokazati da su tačke P, M i Q kolinearne.

Kako je

$$\left.\begin{array}{l} \measuredangle MEQ \cong \measuredangle MBP \\ \measuredangle EQM \cong \measuredangle BPM \\ \overline{ME} \cong \overline{MB} \end{array}\right\} \xRightarrow{UUS} \triangle MBP \cong \triangle MEQ$$

$$\Downarrow$$

$$\measuredangle BMP \cong \measuredangle EMQ$$

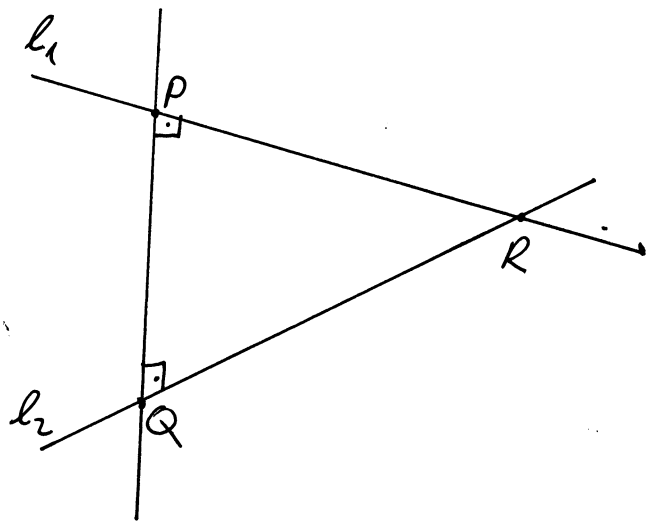
Neka je $R \in \overrightarrow{PM}$ t.d. $P-M-R$. Prema teoremu vertikalnog ugla $\measuredangle BMP \cong \measuredangle EMR$. Time je $\measuredangle EMQ \cong EMR$. Tačke Q i R su sa iste strane prave $\overleftrightarrow{GH} = \ell$ (ZAŠTO?)

Prema teoremu konstrukciji ugla $\measuredangle EMQ = \measuredangle EMR$.

Prema tome $Q \in \operatorname{int}(\overrightarrow{MR}) \subseteq \overleftrightarrow{PM}$ pa su P, M i Q kolinearne.

Time smo dobili da je prava $\overleftrightarrow{PQ}$ okomita na ℓ_1 i ℓ_2.

Sad pretpostavimo da je $\ell_1 \cap \ell_2 = \{R\}$



Tada $P \neq R$, $Q \neq R$ i P, Q i R su nekolinearne. Ali tada $\triangle PQR$ ima dva prava ugla, što nije moguće.

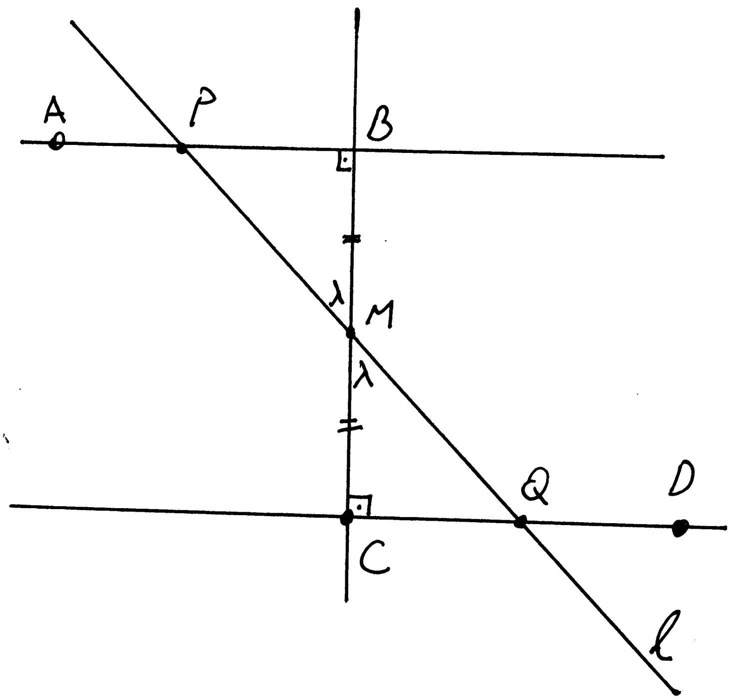
Prema tome $\ell_1 \cap \ell_2 = \emptyset$

a time $\ell_1 \parallel \ell_2$

q.e.d.

(#) U neutralnoj geometriji, ako je $\overleftrightarrow{BC}$ zajednička normala na $\overleftrightarrow{AB}$ i $\overleftrightarrow{CD}$, dokazati da ako je l transferzala pravih $\overleftrightarrow{AB}$ i $\overleftrightarrow{CD}$ koja sadrži sredinu duži $\overline{BC}$ tada je par naizmjeničnih unutrašnjih uglova za pravu l podudaran.

Rj.



Sredinu duži $\overline{BC}$ označimo sa M, a presječne tačke prave l sa pravima $\overleftrightarrow{AB}$ i $\overleftrightarrow{CD}$ označimo redom sa P i Q.

Kako su $\measuredangle PMB$ i $\measuredangle QMC$ unakrsni uglovi to je

$$\measuredangle PMB \cong \measuredangle QMC$$

Sad imamo

$$\left.\begin{array}{c} \measuredangle PMB \cong \measuredangle QMC \\ \overline{MB} \cong \overline{MC} \\ \measuredangle MBP \cong \measuredangle MCQ \end{array}\right\} \overset{USU}{\Longrightarrow} \triangle PMB \cong \triangle QMC$$

$$\Downarrow$$

$$\measuredangle BPM \cong \measuredangle CQM$$

q.e.d.